7.4 Midpoints and Medians in Triangles

In this section, you will examine the properties of line segments that divide triangles in various ways. These properties are useful in calculations for the design of buildings and machinery.

Investigate

Tools

- ruler
- protractor

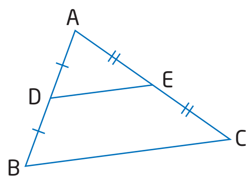
midpoint

- the point that divides a line segment into two equal segments

What are the properties of the midpoints of the sides of a triangle?

Method 1: Use Paper and Pencil

1. Draw a large triangle on a sheet of paper. Label the vertices A, B, and C. Then, measure the length of side AB and mark the **midpoint**. Label this point D. Find the midpoint of AC and label it E. Draw a line segment from D to E.



2. Measure the lengths of DE and BC. How are these lengths related?

3. If the co-interior angles formed by a transversal and two line segments are supplementary, the two segments are parallel. Determine whether DE is parallel to BC.

4. Fold your diagram across the line through points D and E. Where does the vertex A touch the lower part of the diagram?

5. What can you conclude about the heights of $\triangle ADE$ and $\triangle ABC$? How is the height of $\triangle ADE$ related to the height of quadrilateral BCED?

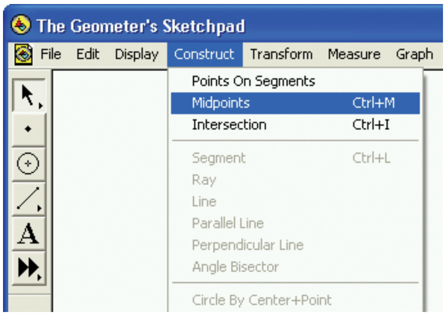
6. Compare your results from steps 2 to 5 with your classmates' results.

7. **Reflect** Do you think your results apply for all triangles? Explain your reasoning.

Method 2: Use *The Geometer's Sketchpad*®

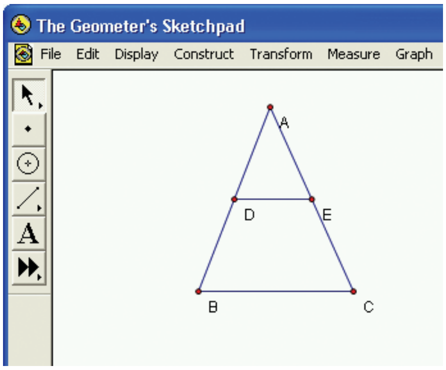
1. Turn on automatic labelling of points. From the **Edit** menu, choose **Preferences**. Click on the **Text** tab, check **For All New Points**, and click on **OK**.

2. Construct a △ABC. To construct the midpoints of AB and AC, select these two sides (but not the vertices), and choose **Midpoints** from the **Construct** menu.



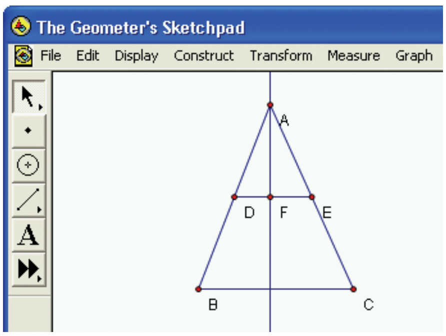
3. Measure the lengths of DE and BC. How are these lengths related?

4. If the co-interior angles formed by a transversal and two line segments are supplementary, the two segments are parallel. Use the sum of ∠EDB and ∠DBC to determine whether DE is parallel to BC.



5. Select vertex A and side BC. From the **Construct** menu, choose **Perpendicular Line**. Select the perpendicular line and line segment DE. Then, choose **Intersection** from the **Construct** menu.

6. Compare the height of △ADE to the height of △ABC.



7. Compare the height of △ADE to the height of quadrilateral BCED.

8. Watch the length and angle measures as you drag vertex A to various new locations. Do any of the length ratios change? Does the sum of ∠EDB and ∠DBC remain constant? Try dragging vertices B and C around the screen as well.

9. **Reflect** What properties does the line segment joining the midpoints of two sides of a triangle have?

Tools

- computer with *The Geometer's Sketchpad*®

Technology Tip

The keyboard shortcut for choosing **Midpoints** is Ctrl+M.

Technology Tip

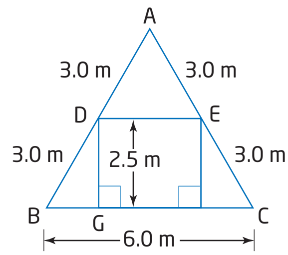
You can also do this investigation using Cabri Jr. on a graphing calculator. For step-by-step instructions, follow the links at www.mcgrawhill.ca/links/principles9.

Example 1 A-Frame Construction

In areas that get a lot of snow, cottages are often built with a triangular shape called an A-frame. This shape helps prevent damage from heavy loads of snow on the roof.

a) Find the width of the floor of the upper room in this cottage.

b) Find the height of the upper room.



Solution

a) Since BD = DA, point D is the midpoint of side AB. Similarly, point E is the midpoint of side AC.

From the properties of midpoints of the sides of a triangle, DE must be half the length of BC. On the drawing, the length of BC is 6.0 m. So, the floor of the upper room is 3.0 m wide.

b) Since D and E are midpoints of two sides of △ABC, the height of △ADE is equal to the height of trapezoid DECB.

The height of the upper room is 2.5 m.

median

- the line segment joining a vertex of a triangle to the midpoint of the opposite side

bisect

- divide into two equal parts

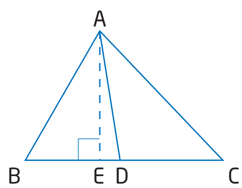
Example 2 Medians of a Triangle

Show that a **median** **bisects** the area of a triangle.

Solution

Method 1: Use the Area Formula

The median AD joins the vertex A to the midpoint of CB. Therefore, CD = BD.



The formula for the area of a triangle is $A = \frac{bh}{2}$.

Since CD = BD, the bases of △ACD and △ADB are equal. These two triangles also have the same height, shown by altitude AE. The areas of the two triangles are equal. Therefore, the median AD divides the area of △ABC into two equal parts.

The same logic applies to a median drawn from any vertex of any triangle. Thus, any median of a triangle bisects its area.

Method 2: Use *The Geometer's Sketchpad®*

Construct any △ABC. Construct the midpoint of side BC by selecting the side and choosing **Midpoints** from the **Construct** menu. Construct a line segment from this midpoint to vertex A. This line segment is a median.

Select points A, B, and D; then, choose **Triangle Interior** from the **Construct** menu. Select points A, D, and C; then, choose **Triangle Interior** from the **Construct** menu again.

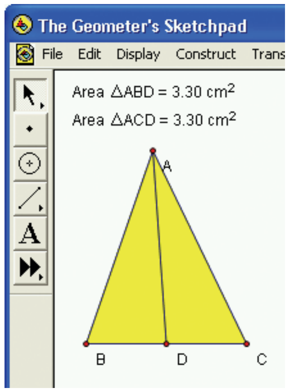
Select the interior of △ABD, and choose **Area** from the **Measure** menu. Measure the area of △ACD in the same way. These measures show that the two areas are equal.

Drag any of the vertices A, B, and C around the screen. The software automatically moves point D so that it stays at the midpoint of BC and AD is still a median. The areas of △ACD and △ABD remain equal for all shapes of △ABC.

This relationship shows that a median bisects the area of any triangle.

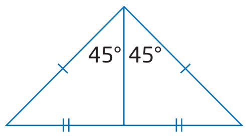
Tools

- computer with *The Geometer's Sketchpad®*



Example 3 Use a Counter-Example

Shivany measured this right triangle and noticed that a median bisects the right angle. She conjectures that a median will bisect the right angle in all right triangles. Is this conjecture correct?

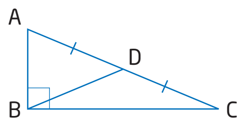


Literacy Connections

A conjecture is an educated guess.

Solution

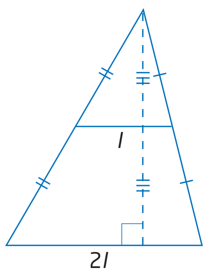
In this right triangle, ∠ABD and ∠DBC are clearly not equal. Thus, the median does not bisect the right angle.



This **counter-example** shows that Shivany's conjecture is incorrect.

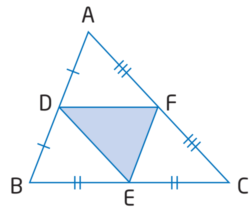
Key Concepts

- A line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long.
- The height of a triangle formed by joining the midpoints of two sides of a triangle is half the height of the original triangle.
- The medians of a triangle bisect its area.
- A counter-example can disprove a conjecture or hypothesis.



Communicate Your Understanding

C1 Points D, E, and F are the midpoints of the sides of $\triangle ABC$. Explain how you can show that the area of $\triangle DEF$ is one quarter of the area of $\triangle ABC$.



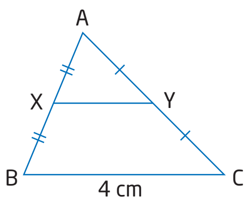
C2 Explain how you could use a counter-example to disprove the hypothesis that all scalene triangles are acute.

Practise

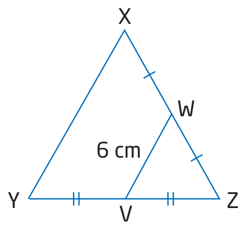
For help with question 1, see Example 1.

1. Calculate the length of line segment XY in each triangle.

a)



b)

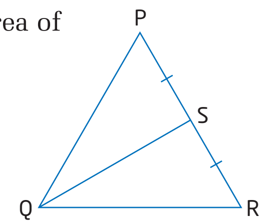


For help with questions 2 and 3, see Example 2.

2. The area of $\triangle PQR$ is 16 cm^2. Calculate the area of

a) $\triangle PQS$

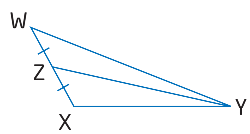
b) $\triangle QSR$



3. The area of $\triangle XYZ$ is 19 cm^2. Calculate the area of

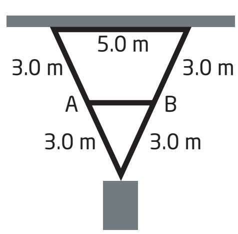
a) $\triangle WZY$

b) $\triangle WXY$



Connect and Apply

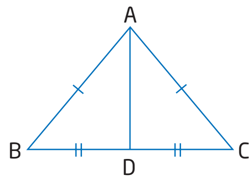
4. Calculate the length of the cross-brace AB in this bridge support.

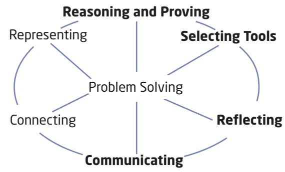


Did You Know?

The cross-brace stops the weight of the bridge from bending the sides of the support outward.

5. a) Make a conjecture about whether the median to the vertex opposite the unequal side of an isosceles triangle bisects the angle at the vertex.



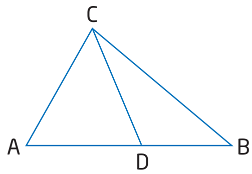


b) Describe how you can see if your conjecture is correct by folding a diagram of an isosceles triangle.

c) Describe how you could use geometry software to see if your conjecture is correct.

d) Use one of the two methods you described to test your conjecture. Describe your results.

6. Raquel conjectures that $\angle ADC$ in this diagram will be acute when point D is located anywhere on side AB. Use a counter-example to show that this conjecture is false.



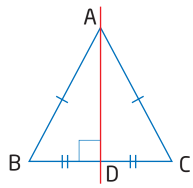
7. Here are three conjectures about scalene triangles with a 60° interior angle. For each conjecture, either draw a counter-example or explain why you think the conjecture is true.

a) The 60° angle is always opposite the shortest side.

b) The 60° angle is always opposite the longest side.

c) The 60° angle is always opposite the second-longest side.

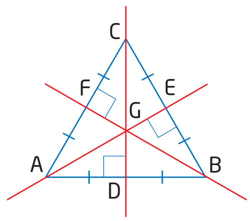
8. Harpreet constructed $\triangle ABC$ with $AB = AC$. He then constructed the midpoint of BC at D and drew a perpendicular line through BC at D. Will this **right bisector** pass through vertex A? Justify your answer.



right bisector

- a line perpendicular to a line segment and passing through its midpoint

9. Tori constructed an equilateral $\triangle ABC$ and the right bisector of each side. She found that the three bisectors intersect at point G. Tori conjectured that $\triangle AGC$, $\triangle CGB$, and $\triangle BGA$ are also equilateral triangles. Is she correct? Explain.



10. **Chapter Problem** Determine whether the three medians of a triangle intersect at a single point.

- If you are using pencil and paper, draw the medians in at least one example of each type of triangle.
- If you are using geometry software, construct a triangle and line segments joining each vertex to the midpoint of the opposite side. Drag each vertex to various new locations. Does changing the shape of the triangle affect how the medians intersect?

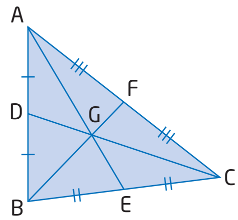
Do you think that the medians intersect at a single point in all triangles? Explain your reasoning.

Extend

centroid

- the point where the medians of a triangle intersect

11. The three medians of this triangle intersect at point G. This point is called a **centroid**.



a) Show that △BEG has the same area as △CEG.

b) Can you use your answer to part a) to show that the area of △ADG is equal to the area of △BDG and that the area of △AFG is equal to the area of △CFG? Explain.

c) Show that all six of the triangles in part a) and part b) have the same area.

similar

- having all corresponding sides proportional

Go to www.mcgrawhill.ca/links/principles9 and follow the links to learn more about Sierpinski's triangle.

12. The Polish mathematician Waclaw Sierpinski devised a process for repeatedly dividing a triangle into smaller **similar** triangles. Question C1 on page 398 shows the first step in this process.

a) Use a library or the Internet to learn how to produce Sierpinski's triangle.

b) Conjecture what fraction of the triangle is shaded after each step of the process.

c) Calculate the area that is shaded after four steps.

13. a) Investigate whether the right bisectors of the sides of a triangle always intersect at a single point. Describe your findings.

b) Draw a triangle in which the right bisectors of the sides intersect at a single point. Can you draw a circle that has this point as its centre and intersects the triangle at exactly three points? If so, describe the properties of the circle.

14. a) Investigate whether the lines that bisect the angles of a triangle always intersect at a single point. Describe your findings.

b) Draw a triangle in which the angle bisectors intersect at a single point. Can you draw a circle that has this point as its centre and intersects the triangle at exactly three points? If so, describe the properties of the circle.

15. Math Contest Is the intersection of the right bisectors of the sides of a triangle always inside the triangle? Support your answer with a diagram.

16. Math Contest Which of these ratios cannot represent the relative lengths of the sides of a triangle?

a) 1:1:1 **b)** 1:2:2 **c)** 1:2:3 **d)** 1:1:2

e) 3:4:5 **f)** 3:4:6 **g)** 3:4:8